

NON-COMMUTATIVE LAURENT PHENOMENON FOR TWO VARIABLES

ALEXANDR USNICH

ABSTRACT. We prove the non-commutative Laurent phenomenon for two variables

1. INTRODUCTION

Let us consider an automorphism of the field $K = \mathbf{C}(x, y)$ given by the formula:

$$F : (x, y) \mapsto \left(\frac{H(x)}{y}, x \right),$$

where $H(x) = 1 + h_1x + \dots + h_{n-1}x^{n-1} + x^n$ is a reversible polynomial, i.e. $h_i = h_{n-i}$.

The iterations of F are actually given by Laurent polynomials [7]. It means that for any integer k we have:

$$F^k : (x, y) \mapsto (L_1(x, y), L_2(x, y)),$$

where $L_1, L_2 \in \mathbf{C}[x, x^{-1}, y, y^{-1}]$ are Laurent polynomials.

We introduce a non-commutative analog of this transformation: consider

$$F_{nc} : (x, y) \mapsto (y^{-1}H(x), y^{-1}xy).$$

We view x, y as elements freely generating the non-commutative algebra A by addition, multiplication and taking inverses of some elements. Namely, we have a ring morphism $\phi : A \rightarrow \mathbf{C}(x, y)$, and we can invert elements a which don't belong to the kernel of ϕ . Then F_{nc} is an automorphism of the algebra A . If we allow to invert only elements x, y , then we will obtain the non-commutative subalgebra $\mathbf{C} \langle x, x^{-1}, y, y^{-1} \rangle \subset A$ which we call the ring of non-commutative Laurent polynomials.

We will prove the following result, conjectured by M.Kontsevich:

Theorem 1.1. *For any integer k and for any reversible polynomial $H(x)$, the transformation F_{nc}^k is given by non-commutative Laurent polynomials.*

We call this *the non-commutative Laurent phenomenon*.

Observe that multiplicative commutator $q = x^{-1}y^{-1}xy$ is preserved by F_{nc} . In the light of deformation quantization, people often consider algebra, where q is a central element. We'd like to emphasize that we impose no such condition.

A special case of this transformation, where $H(x) = 1 + x^n$, turns up in the study of cluster mutations. In the article [5] we prove the special case

of the Laurent phenomenon for $n = 2$ using explicite computations with matrices. In [6] an alternative proof of the Laurent phenomenon for $n = 2$ is given, via a combinatorial path-counting argument. It is moreover proved that the coefficients of Laurent polynomials are positive.

The main idea of our proof of Theorem (1.1) is as follows. First we resolve birational map F^k , namely we construct a sequence of surfaces Y_i and morphisms $\pi_i : Y_i \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$, such that the induced birational maps $F_i = \pi_{i+1}^{-1} \circ F \circ \pi_i : Y_i \rightarrow Y_{i+1}$ extend to natural biregular isomorphisms. Here F is as defined previously in affine coordinates x, y on $\mathbb{P}^1 \times \mathbb{P}^1$. The surface Y_i is constructed as a blow-up of a toric surface Y_i^0 , which is a toric weighted blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$, at $2n$ points situated on the chain of toric divisors. We denote by D_i the chain of rational curves on Y_i which is the strict transform of toric divisors on Y_i^0 . In fact the isomorphism F_i sends the chain D_i to the chain D_{i+1} .

Next, we construct quotient triangulated category

$$\tilde{C}(Y_i) = \tilde{D}(Y_i) / \tilde{D}^1(Y_i),$$

where $\tilde{D}(Y_i)$ is a full subcategory of $D(Y_i)$ the derived category of coherent sheaves on Y_i consisting of objects, which are left orthogonal to \mathcal{O}_{Y_i} . $\tilde{D}^1(Y_i)$ is a full subcategory of $\tilde{D}(Y_i)$ consisting of objects which restrict to 0 at the generic point. We use some properties of the category $\tilde{C}(Y_i)$, which are proved in [4]. Namely this category is generated¹ by one object Q_i , which is the image of the line bundle $\pi_i^* \mathcal{O}(1, 1) \in \tilde{D}(Y_i)$. Moreover, we have:

$$\mathrm{Hom}_{\tilde{C}(Y_i)}(Q_i, Q_i) = A,$$

where A is the non-commutative algebra, containing distinguished elements x, y . The functor $\mathbb{L} F_i^*$ descends to an equivalence of quotient categories $\mathbb{L} F_i^* : \tilde{C}(Y_{i+1}) \rightarrow \tilde{C}(Y_i)$. In (3.1) we write down a specific isomorphism between Q_i and $\mathbb{L} F_i^* Q_{i+1}$ in $\tilde{C}(Y_i)$. This gives us an automorphism F_{nc} of A , which doesn't depend on i :

$$\begin{aligned} F_{nc} : A = \mathrm{Hom}_{\tilde{C}(Y_{i+1})}(Q_{i+1}, Q_{i+1}) &\xrightarrow{\mathbb{L} F_i^*} \mathrm{Hom}_{\tilde{C}(Y_i)}(\mathbb{L} F_i^* Q_{i+1}, \mathbb{L} F_i^* Q_{i+1}) \\ &\downarrow \\ &\mathrm{Hom}_{\tilde{C}(Y_i)}(Q_i, Q_i) = A. \end{aligned}$$

In the Lemma 3.1 we compute this automorphism explicitly:

$$F_{nc} : (x, y) \mapsto (y^{-1} H(x), y^{-1} xy).$$

Therefore we see, that the functor

$$\mathbb{L} \Phi^* = \mathbb{L} F_0^* \circ \cdots \circ \mathbb{L} F_{k-1}^* : \tilde{C}(Y_k) \rightarrow \tilde{C}(Y_0)$$

¹by shifts and taking cones

together with the appropriate isomorphism of objects $\Phi^*(Q_k)$ and Q_0 in $\tilde{C}(Y_0)$ induces an automorphism F_{nc}^k of A :

$$A = \text{Hom}_{\tilde{C}(Y_k)}(Q_k, Q_k) \rightarrow \text{Hom}_{\tilde{C}(Y_0)}(Q_0, Q_0) = A.$$

Then we observe, that morphisms $F_{nc}^k(x), F_{nc}^k(y) : Q_0 \rightarrow Q_0$ descend from morphisms in the quotient category $\tilde{D}(Y_0)/\tilde{D}_{D_0}(Y_0)$ (observe $\tilde{D}_{D_0}(Y_0) \subset \tilde{D}^1(Y_0)$), where $\tilde{D}_{D_0}(Y_0)$ is the subcategory of objects supported on the chain of rational curves D_0 . Therefore $F_{nc}^k(x), F_{nc}^k(y)$ can be viewed also as elements in the endomorphism algebra of $\pi_0^*O(1, 1)$ in the quotient category $\tilde{D}(Y_0)/\tilde{D}_B(Y_0)$, where B is the union of D_0 and $2n$ exceptional curves of the blow-up of Y_0^0 .

Finally we prove in the Lemma 3.3 that the image of the natural functor

$$\text{Hom}_{\tilde{D}(Y_0)/\tilde{D}_B(Y_0)}(Q_0, Q_0) \rightarrow \text{Hom}_{\tilde{C}(Y_0)}(Q_0, Q_0) = A$$

is the subalgebra of non-commutative Laurent polynomials

$$\mathbf{C} \langle x, x^{-1}, y, y^{-1} \rangle \subset A.$$

As we observed, $F_{nc}^k(x), F_{nc}^k(y)$ belong to this subalgebra, so they are non-commutative Laurent polynomials.

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2. RESOLUTION OF AUTOMORPHISM

The results of this section appear in [3] in greater generality. We summarize them here for the convenience of our reader.

Consider the birational automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ given by the formula:

$$F : (x, y) \mapsto \left(\frac{H(x)}{y}, x \right),$$

where $H(y) = 1 + h_1x + \dots + h_{n-1}x^{n-1} + x^n$ is a polynomial of degree n . In the homogeneous coordinates $(X : Z) \times (Y : W)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ the affine coordinates are expressed as $x = \frac{X}{Z}, y = \frac{Y}{W}$.

We are interested in constructing explicitly rational surfaces Y_0, \dots, Y_k equipped with morphisms $\pi_i : Y_i \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ and with biregular isomorphisms $F_i : Y_i \rightarrow Y_{i+1}$, such that the following diagrams commute:

$$(2.1) \quad \begin{array}{ccc} Y_i & \xrightarrow{F_i} & Y_{i+1} \\ \pi_i \downarrow & & \downarrow \pi_{i+1} \\ \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{F} & \mathbb{P}^1 \times \mathbb{P}^1 \end{array} ,$$

Let us define two series of vectors in \mathbf{Z}^2 by a recursive relation:

$$p_0 = (0, 1), p_1 = (-1, 0), p_{i+1} = np_i - p_{i-1};$$

$$t_0 = (1, 0), t_1 = (0, -1), t_{i+1} = nt_i - t_{i-1}.$$

Consider toric surfaces Y_i^0 given by the fan spanned by vectors:

$$\{p_i, \dots, p_0, t_0, t_1, t_2, \dots, t_{n+2-i}\}.$$

Surface Y_i is constructed as a blow-up of the surface Y_i^0 in $2n$ points. Fans of surfaces Y_i^0 contain sub-fan $\{p_1, p_0, t_0, t_1\}$, which defines a surface $\mathbb{P}^1 \times \mathbb{P}^1$, so they admit natural toric projections to it. We can actually think of them as weighted blow-ups of $\mathbb{P}^1 \times \mathbb{P}^1$. We use standard notations (x, y) for the coordinates on toric surfaces. Namely if a vector (a, b) corresponds to a toric divisor, then rational function $\frac{x^b}{y^a}$ induces a canonical (up to taking an inverse) rational coordinate on this divisor. By a canonical coordinate on a divisor D we will mean a rational function, which induces an isomorphism of D with \mathbb{P}^1 . On each surface Y_i^0 toric divisors form a chain of rational curves. Their strict transforms form a chain of rational curves on the blow-up Y_i and the canonical rational coordinates lift from each curve to its strict transform.

The toric divisors corresponding to vectors t_i and p_i will be denoted T_i and P_i respectively. Let x be the canonical coordinate on P_0 , and y the canonical coordinate on T_0 . Note that intersection points with other toric divisors have coordinates 0 and ∞ .

We begin with lemma, which shows how to resolve birational transformation: $(x, y) \mapsto (\frac{H(x)}{y}, x)$. Let Z_1^0 be the toric surface corresponding to the fan: $\{p_1, p_0, t_0, t_1, t_2\}$, and let Z_2^0 be the toric surface corresponding to the fan $\{p_2, p_1, p_0, t_0, t_1\}$.

$$(2.2) \quad \begin{array}{ccc} Z_1 & \xrightarrow{G} & Z_2 \\ \downarrow & & \downarrow \\ Z_1^0 & \xrightarrow{\quad} & Z_2^0 \\ \downarrow r_1 & & \downarrow r_2 \\ P^1 \times P^1 & \xrightarrow{F} & P^1 \times P^1 \end{array}$$

The surface Z_1 is a blow-up of Z_1^0 in the n points on curve P_0 , where $H(x) = 0$. The surface Z_2 is a blow-up of Z_2^0 in the n points on T_0 where $H(y) = 0$.

Lemma 2.1. *For any reversible polynomial H with distinct roots, the induced map G is a regular isomorphism of surfaces Z_1, Z_2 . Moreover it preserves the canonical coordinates on the chain of strict transforms of toric divisors.*

Proof. We denote by C_{ab} the cone in \mathbf{R}^2 spanned by vectors a, b . Such cones correspond to toric points, and we have coordinates in the neighbourhood of these points on Z_1^0 .

The coordinates near toric point C_{p_1, p_0} on Z_1^0 are (x^{-1}, y) ;
 near C_{p_0, t_0} are (x, y) ;
 near C_{t_0, t_1} are (x, y^{-1}) ;
 near C_{t_1, t_2} are $(x^{-1}, x^n y^{-1})$;
 and near C_{t_2, p_1} , which is a singular toric point, are $(x^{-1}, y^{-1}, x^{-n} y)$.

When we blow-up surface Z_1^0 at n points on P_0 , we pull-back coordinates near toric points to Z_1 , so the coordinates near pull-back of C_{p_1, p_0} on Z_1 are $(x^{-1}, \frac{y}{H(x^{-1})})$;
 near pull-back of C_{p_0, t_0} are $(x, \frac{y}{H(x)})$.

The coordinates near other pull-backs are the same as on Z_1^0 .

Birational transformation F of $P^1 \times P^1$ lifts to a birational map $F^0 : Z_1^0 \rightarrow Z_2^0$. Under this map toric divisors P_1, P_0, T_0, T_1, T_2 go to divisors P_2, P_1, P_0, T_0, T_1 respectively. We now prove that this map is regular everywhere except at n points on the divisor P_0 where $H(x) = 0$. Because H has distinct roots, all these points are different.

To avoid confusion, we denote by (u, v) the rational coordinates on Z_2^0 and on Z_2 , so that $G^* u = \frac{H(x)}{y}, G^* v = x$. The map G sends the neighbourhood of the point C_{p_2, p_1} to the neighbourhood of the point C_{p_1, p_0} :

$$G^* : \mathbb{C}[u^{-1} v^n, v^{-1}] \rightarrow \mathbb{C}[x^{-1}, \frac{y}{H(x^{-1})}],$$

$$G^*(u^{-1} v^n, v^{-1}) = (\frac{x^n y}{H(x)}, x^{-1}) = (\frac{y}{H(x^{-1})}, x^{-1}).$$

It is an isomorphism of affine neighbourhoods. The canonical coordinate $u^{-1} v^n$ of P_2 on Z_2 goes to $\frac{y}{H(x^{-1})}$, which is equal to y on P_1 , because divisor P_1 is defined by $x^{-1} = 0$. The canonical coordinate v^{-1} on P_1 goes to the canonical coordinate x^{-1} on P_0 .

We do similar verifications for other pull-backs of toric points. For the neighbourhood of $G^*(C_{p_1, p_0}) = C_{p_0, t_0}$ we have:

$$G^* : \mathbb{C}[u^{-1}, v] \rightarrow \mathbb{C}[x, \frac{y}{H(x)}],$$

$$G^*(u^{-1}, v) = (\frac{y}{H(x)}, x).$$

It is again an isomorphism of affine neighbourhoods, and the canonical coordinate u^{-1} on P_0 goes to $\frac{y}{H(x)}$, which is equal to y on T_0 , because T_0 is defined by $x = 0$ in this neighbourhood.

For the neighbourhood of $G^*(C_{p_0, t_0}) = C_{t_0, t_1}$ we have:

$$G^* : \mathbb{C}[\frac{u}{H(v)}, v] \rightarrow \mathbb{C}[x, y^{-1}],$$

$$G^*(u, v) = (\frac{H(x)}{yH(x)}, x) = (y^{-1}, x).$$

It is an isomorphism of affine neighbourhoods. The canonical coordinate v on T_0 goes to x on T_1 . For the neighbourhood of $G^*(C_{t_0,t_1}) = C_{t_1,t_2}$ we have:

$$G^* : \mathbf{C}[\frac{u}{H(v^{-1})}, v^{-1}] \rightarrow \mathbf{C}[x^{-1}, x^n y^{-1}],$$

$$G^*(\frac{u}{H(v^{-1})}, v^{-1}) = (\frac{H(x)}{yH(x^{-1})}, x^{-1}) = (x^n y^{-1}, x^{-1}).$$

It is an isomorphism of affine neighbourhoods. The canonical coordinate u on T_1 goes to $\frac{H(x)}{y}$ on T_2 , but T_2 is defined by $x^{-1} = 0$, so we can write $\frac{H(x)}{y} = (x^n y^{-1})H(x^{-1}) = x^n y^{-1}$. This proves that map G preserves canonical coordinates on toric divisors.

The four neighbourhoods that we considered provide the covering of Z_1 except at the point C_{t_2,p_1} and at n points, each lying on the exceptional curve of the blow-up. We verify, that at these points G is also an isomorphism.

For the neighbourhood of $G^*(C_{t_1,t_2}) = C_{t_2,p_1}$ we have:

$$F^* : \mathbf{C}[u^{-1}, v^{-1}, uv^{-n}] \rightarrow \mathbf{C}[x^{-1}, y^{-1}, x^{-n}y],$$

$$F^*(u^{-1}, v^{-1}, uv^{-n}) = (\frac{y}{H(x)}, x^{-1}, \frac{H(x)}{x^n y}) = ((x^{-n}y)H(x^{-1})^{-1}, H(x^{-1})y^{-1}).$$

This map is well defined outside the divisor $H(x^{-1}) = 0$. The point C_{t_2,p_1} doesn't belong to this divisor, so G is well defined at this point.

If λ is a root of polynomial H , then we have coordinates $(\frac{x-\lambda}{y}, y)$ near the point on the exceptional curve, where we have to verify that G is regular. The coordinates near the corresponding point on Z_2 are $(u, \frac{v^{-1}-\lambda^{-1}}{u})$. It is straitforward to see that G^* defines an isomorphism of local rings. \square

Recall that we defined the toric surface Y_i^0 as given by the fan

$$\{p_i, \dots, p_1, p_0, t_0, t_1, \dots, t_{n+1-i}\}.$$

Let's blow it up at n points where P_0 intersects $H(x) = 0$, and at n points where T_0 intersects $H(y) = 0$. Here x and y are the canonical coordinates on P_0 and T_0 respectively. The canonical coordinate are defined up to an inverse, so the polynomial H needs to be reversible, for the blow up not to depend on the choice of a coordinate. Let us denote this blow-up by Y_i . Let $D_i \subset Y_i$ be the strict transform of toric divisors under this blow-up.

As a corollary of Lemma 2.1 we have:

Lemma 2.2. *If the polynomial H has distinct roots and is reversible, then the map F induces a regular automorphism F_i between Y_i and Y_{i+1} . Moreover $F_i(D_i) = D_{i+1}$.*

Proof. Note that the surface Y_i can be obtained from the surface Z_1 of previous lemma, by making two kinds of blow-ups. First, we perform weighted blow-ups to introduce toric divisors $P_{i+1}, \dots, P_2, T_3, \dots, T_{n+1-i}$. Then we to blow-up n points on the divisor T_0 , defined by the equation $H(y) = 0$. We blow-up Z_2 in a similar fashion to obtain Y_{i+1} . By Lemma 2.1, the map F lifts to regular map G from Z_1 to Z_2 . Divisor P_3 on Z_2 is a weighted blow-up at the point C_{t_1, p_2} . The weights are determined using expression of the vector p_3 as the linear combination of t_1 and p_2 . But this expression is the same as the expression of the vector p_2 as the linear combination of t_2 and p_1 . So G sends the weighted blow-up corresponding to P_3 to the weighted blow-up corresponding to P_2 . The same argument works for other toric divisors P_a, T_b . The toric divisors P_{i+1}, \dots, P_2 are then mapped to P_{i+2}, \dots, P_3 , as well as T_3, \dots, T_{n+1-i} are mapped to T_2, \dots, T_{n-i} .

Also n points on T_0 where $H(y) = 0$ are mapped to n points on P_0 where $H(x) = 0$, because the canonical coordinates are preserved by G by the previous lemma. Therefore, the blow-ups we do to Z_1 to produce Y_i correspond under isomorphism G precisely to the blow-ups we do to Z_2 to produce Y_{i+1} , and hence G lifts to an isomorphism $F_i : Y_i \rightarrow Y_{i+1}$. The last statement of lemma is also clear. \square

This lemma implies, that we have a regular isomorphism of surfaces:

$$\Phi = F_{k-1} \circ \dots \circ F_0 : Y_0 \rightarrow Y_k.$$

3. DG-CATEGORY ASSOCIATED TO A RATIONAL SURFACE

Let $D(Y_i)$ denote the bounded derived category of coherent sheaves on Y_i . By Lemma 2.2 we have a functor

$$\mathbb{L} F_i^* : D(Y_{i+1}) \xrightarrow{\sim} D(Y_i),$$

which is an equivalence of triangulated categories.

In [4] we've introduced the notion of $\tilde{D}(Y_i)$ full triangulated subcategory of $D(Y_i)$ which consists of objects E for which $\mathrm{RHom}_{Y_i}(E, \mathcal{O}_{Y_i}) = 0$. As

$$\mathbb{L} F_i^* \mathcal{O}_{Y_{i+1}} = \mathcal{O}_{Y_i},$$

$\mathbb{L} F_i^*$ restricts to an equivalence

$$\mathbb{L} F_i^* : \tilde{D}(Y_{i+1}) \rightarrow \tilde{D}(Y_i).$$

Let $\pi_i : Y_i \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ be the natural projections. Recall that $\tilde{D}(\mathbb{P}^1 \times \mathbb{P}^1)$ is generated by three objects:

$$\tilde{D}(\mathbb{P}^1 \times \mathbb{P}^1) = \langle \mathcal{O}(1, 0), \mathcal{O}(0, 1), \mathcal{O}(1, 1) \rangle.$$

Denote by $Q_i = \pi_i^* \mathcal{O}(1, 1) \in \tilde{D}(Y_i)$ the pull-back of the line bundle $\mathcal{O}(1, 1)$ by π_i .

Let $D^1(Y_i)$ be the full subcategory of $D(Y_i)$ consisting of objects whose support is at most a divisor, in other words we take objects of $D(Y_i)$ which

restrict to 0 at the generic point of Y_i . Let $D_{D_i}^1(Y_i)$ be the subcategory of $D^1(Y_i)$ consisting of objects supported on D_i , the union of all divisors T_a and P_b .

Observe that $\mathbb{L}F_i^*$ takes subcategories $D_{D_{i+1}}^1(Y_{i+1})$ and $D^1(Y_{i+1})$ to subcategories $D_{D_i}^1(Y_i)$ and $D^1(Y_i)$ respectively. This is because F_i is a regular isomorphism, and $F(D_i) = D_{i+1}$. Let also:

$$\begin{aligned}\tilde{D}^1(Y_i) &= D^1(Y_i) \cap \tilde{D}(Y_i), \\ \tilde{D}_{D_i}(Y_i) &= D_{D_i}^1(Y_i) \cap \tilde{D}(Y_i).\end{aligned}$$

The non-commutative cluster mutations appear, when we look at the factor category

$$\tilde{C}(Y_i) = \tilde{D}(Y_i) / \tilde{D}^1(Y_i).$$

It is proved in [4], that this category is a birational invariant of a variety. For rational surfaces it is generated by one object Q_i , and moreover

$$\mathrm{Hom}_{\tilde{C}(Y_i)}(Q_i, Q_i) = A,$$

where A is a non-commutative algebra. This algebra is a natural setting for non-commutative cluster mutations. Let us recall some properties of this algebra A . First of all there is an embedding $i : \mathbf{C} \langle x, y \rangle \hookrightarrow A$, and there is a natural map $\phi : A \rightarrow \mathbf{C}(x, y)$. Moreover the kernel of the map ϕ is a commutator ideal of A :

$$\ker(\phi) = A[A, A].$$

We also have the following property: any $a \in A$ with $\phi(a) \neq 0$ is invertible.

We now choose a way to identify objects Q_i and F^*Q_{i+1} in $\tilde{C}(Y_i)$. This will induce a map on endomorphism ring of object, so we will get a map $F_{nc} : A \rightarrow A$, which we will compute explicitly.

Recall, that F induces a regular map from Z_1 to Z_2 , and we lift it after making some blow-ups to a regular map from Y_i to Y_{i+1} . So we can choose an identification of $O_{Z_1}(1, 1)$ and $G^*O_{Z_2}(1, 1)$ on Z_1 in $\tilde{C}(Z_1)$, and then lift this identification to $\tilde{C}(Y_i)$.

The surface Z_2 is the blow-up of toric surface Z_2^0 at n points on the toric divisor T_0 . We identify this divisor with its strict transform. Denote by E the exceptional curve of this blow-up. It is the union of n rational curves. Also Z_2 has a chain of rational curves P_2, P_1, P_0, T_0, T_1 . And we have linear equivalences of divisors:

$$O_{Z_2}(0, 1) = P_0 = T_1 + P_2,$$

$$O_{Z_2}(1, 0) = T_0 + E = P_1 + nP_2.$$

The divisor $O_{Z_2}(1, 1)$ is therefore linearly equivalent to $T_1 + P_1 + (n+1)P_2$. Then we compute its pull-back by G to Z_1 :

$$G^*O_{Z_2}(1, 1) = G^*(T_1 + P_1 + (n+1)P_2) = T_2 + P_0 + (n+1)P_1.$$

On Z_1 we have a chain of rational curves P_1, P_0, T_0, T_1, T_2 , and we have the exceptional curve C of the blow-up of P_0 at n points. Note that the

effective divisor $O_{Z_1}(1, 1)(-C) = P_0 + P_1 + T_2$ is dominated by $G^*O_{Z_2}(1, 1) = T_2 + P_0 + (n+1)P_1$, so we have a natural morphism of line bundles

$$i_0 : O_{Z_1}(1, 1)(-C) \xrightarrow{nP_1} G^*O_{Z_2}(1, 1).$$

There is also a unique up to scalar multiplication map of line bundles $i_1 : O_{Z_1}(1, 0) \xrightarrow{P_0} O_{Z_1}(1, 1)(-C)$, which lifts the map $O(1, 0) \rightarrow O(1, 1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$, which vanishes along the divisor P_0 . Finally there is a map $i_3 : O_{Z_1}(1, 0) \xrightarrow{T_1+nT_2} O_{Z_1}(1, 1)$, which vanishes along the divisor T_1 . All in all, we have the following sequence of maps of line bundles on Z_1 :

$$(3.1) \quad G^*O_{Z_2}(1, 1) \xleftarrow{i_1} O_{Z_1}(1, 1)(-C) \xleftarrow{i_2} O_{Z_1}(1, 0) \xrightarrow{i_3} O_{Z_1}(1, 1).$$

If we consider the line bundles in this diagram as objects of the derived category of coherent sheaves $D(Z_1)$ then they belong to $\tilde{D}(Z_1)$. If we pull (3.1) back to Y_i , the objects will belong to $\tilde{D}(Y_i)$. We now claim that the cones of the morphisms in (3.1) belong to $D_{D_i}^1(Y_i)$. Indeed, on the surface Z_1 the object $\text{Cone}(i_1)$ is supported on P_1 , $\text{Cone}(i_2)$ is supported on P_0 , $\text{Cone}(i_3)$ is supported on $T_1 \cup T_2$. Blowing up n points on the curve T_0 doesn't change this. Therefore on Y_i all these cones belong to $D_{D_i}(Y_i)$. Consequently, the morphisms in (3.1) become isomorphisms in the quotient categories $\tilde{D}(Y_i)/\tilde{D}_{D_i}(Y_i)$ and $\tilde{C}(Y_i) = \tilde{D}(Y_i)/\tilde{D}^1(Y_i)$. In the later category we thus obtain a particular isomorphism $j_k : Q_i \rightarrow F_i^*Q_{i+1}$. This isomorphism allows us to define an automorphism of the ring A :

$$(3.2) \quad \begin{array}{ccc} F_{nc} : A = \text{Hom}_{\tilde{C}(Y_{i+1})}(Q_{i+1}, Q_{i+1}) & \xrightarrow{\mathbb{L}F_i^*} & \text{Hom}_{\tilde{C}(Y_i)}(F_i^*Q_{i+1}, F_i^*Q_{i+1}) \\ & & \downarrow j_k^* \\ & & A = \text{Hom}_{\tilde{C}(Y_i)}(Q_i, Q_i). \end{array}$$

Lemma 3.1. *The map F_{nc} is given by*

$$F_{nc} : (x, y) \mapsto (y^{-1}H(x), y^{-1}xy)$$

Proof. We first explain, how we identify the algebra A with the endomorphism ring $\text{Hom}_{\tilde{C}(Y_i)}(Q_i, Q_i)$. If $\pi : X \rightarrow Y$ is a blow-up of a surface Y at the smooth point, then $\mathbb{L}\pi^*$ induces a fully faithful embedding, and we have a semiorthogonal decomposition:

$$D(X) = \langle O_E, \mathbb{L}\pi^*D(Y) \rangle,$$

where O_E is a structure sheaf of the exceptional curve E of the blow-up. The similar decomposition works for weighted blow-ups. Functor $\mathbb{L}\pi^*$ then induces equivalences between quotient categories $\tilde{C}(X)$ and $\tilde{C}(Y)$. In particular, for a surface Y_i we use sequence of blow-ups $\pi_i : Y_i \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ to identify $\tilde{C}(Y_i)$ with $\tilde{C} = \tilde{C}(\mathbb{P}^1 \times \mathbb{P}^1)$.

If $(X : Z) \times (Y : W)$ are homogeneous coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$, then diagram

$$(3.3) \quad O(1, 1) \xleftarrow{Z} O(0, 1) \xrightarrow{X} O(1, 1)$$

defines an element in $\text{Hom}_{\tilde{C}}(O(1, 1), O(1, 1))$, which we denote by x . Similarly the diagram

$$O(1, 1) \xleftarrow{W} O(1, 0) \xrightarrow{Y} O(1, 1)$$

defines an element in $\text{Hom}_{\tilde{C}}(O(1, 1), O(1, 1))$, which we denote by y .

In the article [4] we computed, that $\text{Hom}_{\tilde{C}}(\mathbb{P}^2)(O(2)) = A$. If $(X : Y : Z)$ are homogeneous coordinates on \mathbb{P}^2 , then denote by x, y the elements represented by diagrams $O_{\mathbb{P}^2}(2) \xleftarrow{Z} O_{\mathbb{P}^2}(1) \xrightarrow{X} O_{\mathbb{P}^2}(2)$ and $O_{\mathbb{P}^2}(2) \xleftarrow{Z} O_{\mathbb{P}^2}(1) \xrightarrow{Y} O_{\mathbb{P}^2}(2)$ respectively. Consider the toric surface T , given by the fan $(1, 0), (0, -1), (-1, -1), (-1, 0), (0, 1)$. It admits toric projections to both \mathbb{P}^2 and $\mathbb{P}^1 \times \mathbb{P}^1$. We can therefore pull-back both $D(\mathbb{P}^2)$ and $D(\mathbb{P}^1 \times \mathbb{P}^1)$ to $D(T)$ and compare the diagrams there. We observe that on surface T the divisors $O_T(1, 0), O_T(0, 1)$ embed into $O_T(1)$, and the divisor $O_T(1, 1)$ embeds into $O_T(2)$. Moreover, the diagrams that define x, y in $\tilde{C}(\mathbb{P}^1 \times \mathbb{P}^1)$ and $\tilde{C}(\mathbb{P}^2)$ give the same morphisms in $\tilde{C}(T)$, thus identifying $\text{Hom}_{\tilde{C}(\mathbb{P}^1 \times \mathbb{P}^1)}(O(1, 1), O(1, 1))$ with A .

We now compute the action of F_{nc} on A . It is enough to compute the action on elements x, y . First we compute the preimages of line bundles on Z_1 :

$$\begin{aligned} G^*O_{Z_2}(1, 0) &= G^*(P_1 + nP_2) = P_0 + nP_1, \\ G^*O_{Z_2}(0, 1) &= G^*(T_1 + P_2) = P_1 + T_2, \\ G^*O_{Z_2}(1, 1) &= G^*(P_1 + nP_2 + T_1) = P_0 + (n+1)P_1 + T_2. \end{aligned}$$

Next recall that to represent x, y on Z_2 we need the following maps:

$$\begin{aligned} X, Z : O_{Z_2}(0, 1) &\rightarrow O_{Z_2}(1, 1), \\ Y, W : O_{Z_2}(1, 0) &\rightarrow O_{Z_2}(1, 1). \end{aligned}$$

The map X defines an inclusion of line bundles $O_{Z_2}(0, 1) \xrightarrow{T_0+E} O_{Z_2}(1, 1)$, given by divisor $T_0 + E$. Consequently we will write the equality, where both sides are understood as inclusions of line bundles:

$$X = T_0 + E.$$

In a similar way we compute:

$$\begin{aligned} Z &= P_1 + nP_2, \\ Y &= P_0, \\ W &= T_1 + P_2. \end{aligned}$$

Therefore we can compute the pull-backs:

$$\begin{aligned} G^*X &= T_1 + G^*(E), \\ G^*Z &= P_0 + nP_1, \end{aligned}$$

$$G^*Y = T_0,$$

$$G^*W = T_2 + P_1.$$

Let π'_1 be the projection of Z_1 to the toric surface Z_1^0 , and C is the exceptional curve of the blow-up. As before π_1 is the toric projection from Z_1^0 to $\mathbb{P}^1 \times \mathbb{P}^1$. We can write, using the same notation for the toric divisor P_0 on Z_1^0 and for its strict transform to Z_1 :

$$P_0 + C = \pi_1'^* P_0.$$

Moreover for any line bundle L on Z_1^0 we have

$$(\pi_1'^* L)_C = O_C.$$

This implies that there are exact sequences of coherent sheaves on Z_1 :

$$0 \rightarrow G^*O_{Z_2}(1, 0) \rightarrow (\pi_1')^*(P_0 + nP_1) \rightarrow O_C \rightarrow 0,$$

$$0 \rightarrow G^*O_{Z_2}(1, 1) \rightarrow (\pi_1')^*(P_0 + (n+1)P_1 + T_2) \rightarrow O_C \rightarrow 0.$$

Observe that O_C is an object of $\tilde{D}(Z_1)$. The terms on the left and on the right in both sequences belong to the category $\tilde{D}(Z_1)$, therefore so do the terms in the middle.

On the surface Z_1^0 we have:

$$O_{Z_1^0}(1, 0) = P_1 + T_2.$$

Therefore we have inclusions of line bundles on Z_1^0 , which are isomorphisms outside T_2 :

$$P_0 + nP_1 \xrightarrow{nT_2} P_0 + nP_1 + nT_2 = O_{Z_1}(n, 1),$$

$$P_0 + (n+1)P_1 + T_2 \xrightarrow{nT_2} P_0 + (n+1)P_1 + (n+1)T_2 = O_{Z_1}(n+1, 1).$$

We use objects $O(n, 1), O(n+1, 1) \in \tilde{D}(\mathbb{P}^1 \times \mathbb{P}^1)$, and their pullbacks $O_{Z_1}(n, 1), O_{Z_1}(n+1, 1)$ to Z_1 . We have inclusions of line bundles on surface Z_1 :

$$i : G^*O_{Z_2}(1, 0) \xrightarrow{C+nT_2} O(n, 1),$$

$$j : G^*O_{Z_2}(1, 1) \xrightarrow{C+nT_2} O(n+1, 1).$$

And therefore we have compositions

$$j \circ \mathbb{L} G^*X, j \circ \mathbb{L} G^*Z : G^*O_{Z_2}(0, 1) = O_{Z_1}(1, 0) \rightarrow O_{Z_1}(n+1, 1).$$

Now recall that $G^*X = T_1 + G^*E$, $j = C + nT_2$, therefore $j \circ F^*X = C + T_1 + G^*E + nT_2$. But the morphism $j \circ G^*X$ is a lift of a morphism from $\mathbb{P}^1 \times \mathbb{P}^1$, where it is given by

$$(\pi_1) \circ (\pi_1')(C + T_1 + G^*E + nT_2) = T_1 + \pi_1(\pi_1'(G^*E)).$$

The divisor $\pi_1(\pi_1'(G^*E))$ is given by $H(x) = 0$ on $\mathbb{P}^1 \times \mathbb{P}^1$. If we introduce homogeneous polynomial $H(X, Z)$ defined by the condition that $\frac{H(X, Z)}{Z^n} =$

$H(\frac{X}{Z})$, then the inclusion of line bundles $j \circ G^* X : O_{Z_1}(1, 0) \rightarrow O_{Z_1}(n+1, 1)$ is a pullback of the map $H(X, Z)W$. We can write

$$j \circ G^* X = Z^n H(\frac{X}{Z})W.$$

By the similar argument $j \circ G^* Z = (\pi_1 \circ \pi'_1)^*(P_0 + nP_1)$. It follows that $j \circ G^* Z = Z^n Y$.

Both inclusions i and j are given by the same divisor $C + nT_2$, so

$$G^* Y, G^* W \in \text{Hom}(G^* O_{Z_2}(1, 0), G^* O_{Z_2}(1, 1)) = \text{Hom}(O(n, 1), O(n+1, 1)).$$

Inclusion $G^* Y$ is given by T_0 , so $j \circ G^* Y = X \circ i$. Inclusion $G^* W$ is given by $T_2 + P_1$, so $j \circ G^* W = Z \circ i$.

We also need to know the map $j \circ i_1 \circ i_2 : O(1, 0) \rightarrow O(n+1, 1)$, where i_1, i_2 are used in (3.1) to identify $O_{Z_1}(1, 0)$ and $G^* O_{Z_2}(1, 1)$ in $\tilde{C}(Z_1)$.

In our notations $i_3 = W$. By using the similar techniques, we see that on the surface $\mathbb{P}^1 \times \mathbb{P}^1$ we have $j \circ i_1 \circ i_2 = (\pi_1 \circ \pi'_1)^*(P_0 + nP_1)$. It implies that $j \circ i_1 \circ i_2 = Z^n Y$.

We use map Z to identify $O_{Z_1}(l, 1)$ and $O_{Z_1}(l+1, 1)$ in $\tilde{C}(Z_1)$, and the map W to identify $O_{Z_1}(1, l)$ and $O_{Z_1}(1, l+1)$ in $\tilde{C}(Z_1)$.

Let us denote by

$$\alpha \in \text{Hom}_{\tilde{C}(Z_1)}(O_{Z_1}(1, 1), O_{Z_1}(n+1, 1))$$

the following morphism

$$\alpha = j \circ i_1 \circ i_2 \circ i_3^{-1}.$$

Then we can write the element $F_{nc}(x)$ in the category $\tilde{C}(Z_1)$ as:

$$\begin{aligned} F_{nc}(x) &= \alpha^{-1} \circ j \circ F^* X \circ (F^* Z)^{-1} \circ j^{-1} \circ \alpha = \\ &= (Z^n Y)^{-1} Z^n H(\frac{X}{Z})W (Z^n Y)^{-1} (Z^n Y) = y^{-1} H(x) y^{-1} y = y^{-1} H(x). \end{aligned}$$

Similarly

$$F_{nc}(y) = (\alpha)^{-1} \circ j \circ F^* Y \circ (F^* W)^{-1} \circ j^{-1} \circ \alpha = (Z^n Y)^{-1} X Z^{-1} (Z^n Y) = y^{-1} x y.$$

The claim of the lemma follows from the observation, that the pull-back along the map $Y_i \rightarrow Z_1$ induces equivalence of categories $\tilde{C}(Z_1)$ and $\tilde{C}(Y_i)$. In particular we note, that the formula for F_{nc} doesn't depend on i . \square

We can now proceed to the final argument.

Theorem 3.1. $F_{nc}^k(x), F_{nc}^k(y)$ are non-commutative Laurent polynomials.

Proof. First note that elements $F_{nc}^k(x), F_{nc}^k(y) \in A$ are represented by elements of $\text{Hom}_{\tilde{C}(Y_0)}(Q_0, Q_0)$. Let D_i be the chain of strict transforms of toric divisors from Y_i^0 to Y_i . We have natural functor

$$\mathbb{K}_i : \tilde{D}(Y_i) / \tilde{D}_{D_i}(Y_i) \rightarrow \tilde{D}(Y_i) / \tilde{D}^1(Y_i) = \tilde{C}(Y_i).$$

In particular, we have the induced map

$$\mathbb{K}_i : \text{Hom}_{\tilde{D}(Y_i)/\tilde{D}_{D_i}(Y_i)}(Q_i, Q_i) \rightarrow \text{Hom}_{\tilde{C}(Y_i)}(Q_i, Q_i).$$

Lemma 3.2. *Elements $F_{nc}^k(x)$, $F_{nc}^k(y)$ belong to the image of \mathbb{K}_0 .*

Proof. By definition (3.2) of F_{nc} we have:

$$F_{nc}^k = j_0^* \circ \mathbb{L} F_0^* \circ \cdots \circ j_{k-1}^* \circ \mathbb{L} F_{k-1}^*.$$

If $\Phi = F_{k-1} \circ \cdots \circ F_0 : Y_0 \rightarrow Y_k$, and $\sigma : \Phi^* O_{Y_k}(1, 1) \xrightarrow{\sim} O_{Y_0}(1, 1)$ is an appropriate identification in the category $\tilde{C}(Y_0)$, then F_{nc}^k is the composition

$$\text{Hom}_{\tilde{C}(Y_k)}(Q_k, Q_k) \xrightarrow{\mathbb{L}\Phi^*} \text{Hom}_{\tilde{C}(Y_0)}(\Phi^* Q_k, \Phi^* Q_k) \xrightarrow{\sigma} \text{Hom}_{\tilde{C}(Y_0)}(Q_0, Q_0).$$

Observe, that $x = X \circ Z^{-1}$ as defined in (3.3) is well-defined morphism in $\tilde{D}(Y_k)/\langle \text{Cone}(Z) \rangle$, because it uses the inverse of morphism Z . But $\text{Supp}(\text{Cone}(Z)) = P_1 \cup P_2 \cup \cdots \cup P_{k+1} \subset D_k$, so in particular it is an element of $\tilde{D}(Y_k)/\tilde{D}_{D_k}(Y_k)$. Similarly $y = Y \circ W^{-1}$ is well-defined morphism of $\tilde{D}(Y_k)/\tilde{D}_{D_k}(Y_k)$, because $\text{Supp}(\text{Cone}(W)) = T_1 \cup P_2 \cup \cdots \cup P_{k+1} \subset D_k$.

Lemma (2.2) implies, that $\Phi^{-1}(D_k) = D_0$. So $\mathbb{L}\Phi^*(x)$, $\mathbb{L}\Phi^*(y)$ are well-defined in the category $\tilde{D}(Y_0)/\tilde{D}_{D_0}(Y_0)$.

Morphism σ is a composition of morphisms of the kind $\mathbb{L}(F_{i-1} \cdots F_0)^* \circ j_i$. Recall that j_i is defined in (3.2) using identification $i_3 \circ i_2^{-1} \circ i_1^{-1}$ of $F_i^* Q_{i+1}$ and Q_i as in (3.1). Observe, that i_1, i_2, i_3 are invertible isomorphisms in $\tilde{D}(Y_i)/\tilde{D}_{D_i}(Y_i)$, therefore σ is invertible in $\tilde{D}(Y_0)/\tilde{D}_{D_0}(Y_0)$. This proves the lemma. \square

Let us take a curve $B = \pi_0^{-1}(XYZW = 0) \subset Y_0$, which is the preimage of all toric divisors on $\mathbb{P}^1 \times \mathbb{P}^1$. It is the union of strict transform of toric divisors D_0 and $2n$ exceptional curves of blow-up of Y_0^0 . Then we have:

Lemma 3.3. *In the quotient category $C = \tilde{D}(Y_0)/\tilde{D}_B(Y_0)$ we have:*

$$\text{Hom}_C(O(1, 1), O(1, 1)) = \mathbf{C} \langle x, x^{-1}, y, y^{-1} \rangle.$$

Proof. By construction π_0 is a composition of regular maps: $Y_0 \rightarrow Y_0^0 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$, where first arrow is a blow-up at $2n$ distinct smooth points, and Y_0^0 is a toric surface. For the blow-up $\pi : Y_0 \rightarrow Y_0^0$ with exceptional divisor E we have a semiorthogonal decomposition[1], [2]:

$$\tilde{D}(Y_0) = \langle \mathbb{L}\pi^*(\tilde{D}(Y_0^0)), O_E \rangle.$$

So we have an equivalence of categories

$$\tilde{D}(Y_0)/\tilde{D}_B(Y_0) \rightarrow \tilde{D}(Y_0^0)/\tilde{D}(Y_0^0)_{\text{tor}}.$$

In the last formula $\tilde{D}(Y_0^0)_{\text{tor}}$ is the full subcategory of objects supported on toric divisors. Because of the semiorthogonal decomposition of the blow-up, this quotient category is the same for any toric surface. Even though Y_0^0 is not smooth, we can consider a smooth toric surface T with an toric

morphism $f : T \rightarrow Y_0^0$, and we can speak about the quotient $\tilde{D}(T)/\tilde{D}(T)_{tor}$ instead. We didn't do it in order to avoid cumbersome formulas.

As a consequence we have:

$$\begin{aligned} C &= \tilde{D}(Y_0)/\tilde{D}_B(Y_0) = \tilde{D}(\mathbb{P}^1 \times \mathbb{P}^1)/\tilde{D}_{(XYZW=0)}(\mathbb{P}^1 \times \mathbb{P}^1) = \\ &= \langle O(0,1), O(1,0), O(1,1) \rangle / \langle Cone(X), Cone(Y), Cone(Z), Cone(W) \rangle = \\ &= D(\mathbf{C} \langle x, y \rangle -mod) / \langle Cone(x), Cone(y) \rangle . \end{aligned}$$

In the last category we have:

$$\mathrm{Hom}(O(1,1), O(1,1)) = \mathbf{C} \langle x, x^{-1}, y, y^{-1} \rangle .$$

□

We have the following maps

$$\mathrm{Hom}_{\tilde{D}(Y_0)/\tilde{D}_{D_0}(Y_0)}(Q_0, Q_0) \rightarrow \mathrm{Hom}_{\tilde{D}(Y_0)/\tilde{D}_B(Y_0)}(Q_0, Q_0) \rightarrow \mathrm{Hom}_{\tilde{C}(Y_0)}(Q_0, Q_0) = A.$$

Lemma 3.2 implies that $F_{nc}^k(x), F_{nc}^k(y)$ belong to the image of the composition of these maps. In particular, they belong to the image of the second map, which is the subalgebra $\mathbf{C} \langle x, x^{-1}, y, y^{-1} \rangle \subset A$ by Lemma 3.3. This proves the theorem. □

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